Math 3235 Probability Theory

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Markov Inequality.

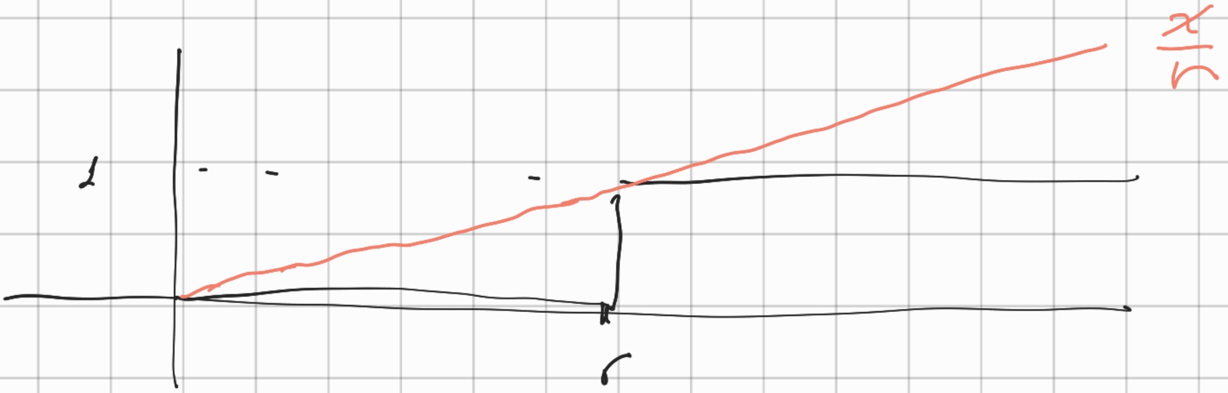
X such that $\mathbb{P}(X > 0) = 1$.

$\mathbb{E}(X) < +\infty$.

$$\mathbb{P}(X > r) \leq \frac{\mathbb{E}(X)}{r}$$

Proof:

$$I_r(x) = \begin{cases} 0 & x < r \\ 1 & \text{otherwise} \end{cases}$$



$$\mathbb{P}(X > r) = \mathbb{E}(I_r(X))$$

$$= \int_r^{\infty} f_X(x)\, dx$$

$$I_r(x) \leq \frac{x}{r}$$

$$\mathbb{P}(X \geq r) \cdot \mathbb{E}(I_r(X)) \leq$$

$$\leq \frac{\mathbb{E}(X)}{r}$$

Chebyschev Inequality

X be a r.v. with $\mathbb{E}(X) = \mu$

$$\mathbb{P}(|X - \mu| > r) \leq$$

$$\frac{var(X)}{r^2}$$

Proof.

$$(X - \mu)^2$$

$$\mathbb{P}(|X - \mu| \geq r) = \mathbb{P}((X - \mu)^2 \geq r^2) \leq$$

$$\leq \frac{\mathbb{E}((X-\mu)^2)}{r^2} =$$

$$= \frac{var(X)}{r^2}$$

$\circ$

Suppose X_i $i = 1 \dots N$

Bernoulli p

$$\bar{X} = \frac{1}{N} \sum_i X_i$$

$\mathbb{E}(X_i) = p$ $\qquad var(X_i) = p(1-p)$

$$\mathbb{E}(\bar{X}) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}(X_i) = \frac{1}{N} \sum_{i=1}^{N} p = p$$

$$var(\bar{X}) = var\left(\frac{1}{N}\sum_{i=1}^{N} X_i\right) = \frac{1}{N^2} var\left(\sum_i X_i\right) =$$

$$= \frac{1}{N^2} \sum_{i=1}^{N} var(X_i) = \frac{p(1-p)}{N}$$

$$\mathbb{P}(|\overline{X} - p| > r) \leq \frac{p(1-p)}{N r^2}$$

$$\overline{X}_N$$

$$\lim_{N \to \infty} \mathbb{P}(|\overline{X}_N - p| > r) = 0 \qquad \forall r$$

$x_1 \quad \dots \quad x_N$

$\overline{x} = \frac{1}{N} \sum_{i=1}^{p} x_i$ is a realization of $\overline{X} = \frac{1}{N} \sum_{i=1}^{N} X_i$

Prob. That $\overline{x}$ is far from p goes To 0 when N goes To infinity.

X_n is a sequence of r.v.

$X_n \to X$ when $n \to \infty$

$a_n \to a$ as $n \to \infty$

$$\lim_{n \to \infty} a_n = a$$

$\forall \varepsilon \; \exists N$ such that $\forall n > N$

$$|a_n - a| < \varepsilon$$

$$e = \lim \left(1 + \frac{1}{N}\right)^N$$

I want an approx. of e with an error of at most 10^{-8}

There exist N such that for all $n > N$

$$\left|\left(1+\frac{1}{n}\right)^n - e\right| < 10^{-8}$$

$$\left(1+\frac{1}{n}\right)^n = e^{n\ln\left(1+\frac{1}{n}\right)}$$

$$\ln\left(1+\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{n^2} + O(n^{-3})$$

$$e^{n\ln\left(1+\frac{1}{n}\right)} = e^{1-\frac{1}{n}+O(n^{-2})} =$$

$$e\left(e^{-\frac{1}{n}+O(n^{-2})}\right) =$$

$$= e\left(1-\frac{1}{n}+O(n^{-2})\right)$$

$|a_n - a|$ distance between a_n and a

$$E\left((X_n - X)^2\right) \geq 0$$

$$E\left((X-Y)^2\right) = 0 \implies P(X-Y=0) = 1$$

Definition:

We say that

$X_n \to X$ in square mean

in L^2

if $\lim_{n\to\infty} \mathbb{E}((X_n - X)^2) = 0$

$X_n \to X$ in square mean

$\forall \varepsilon \ \exists N$ such That if $n > N$

Than $\mathbb{E}((X_n - X)^2) < \varepsilon$

$|a_n - a| < \varepsilon$

Theorem: if X_i are i.i.d. r.v.

(i.i.d. independent and identically distributed)

Calling $\overline{X}_N = \frac{1}{N} \sum_{i=1}^{N} X_i$

we have

$$\overline{X}_N \to \mu \quad \text{in square mean}$$

where $\mathbb{E}(X_i) = \mu$.

Proof:

Since $\mathbb{E}(\overline{X}_N) = \mu$ we have

$$\mathbb{E}\left(\left(\overline{X}_N - \mu\right)^2\right) = Var(\overline{X}_N) =$$

$$= \frac{Var(X_i)}{N}$$